MATH2060A Solution to Assignment 1

Section 6.1

4. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$ for rational x and $f(x) = 0$ for irrational x. Show that f is differentiable at $x = 0$ and find $f'(0)$.

We claim that f is differentiable at 0 with $f'(0) = 0$. Consider the difference quotient

$$\frac{f(x) - f(0)}{x - 0} \quad x \neq 0.$$

When x is rational, it is equal to x and, when x is irrational, it is equal to 0. Therefore,

$$\left| \frac{f(x) - f(0)}{x - 0} - 0 \right| \leq |x| \; .$$

For every $\varepsilon > 0$, we take $\delta = \varepsilon$, then

$$\left| \frac{f(x) - f(0)}{x - 0} - 0 \right| \leq |x| < \varepsilon, \quad x \neq 0, |x| < \delta.$$

We conclude that $f'(0) = 0$.

7. $\dfrac{g(x) - g(c)}{x - c} = \dfrac{|f(x)| - |f(c)|}{x - c} = \text{sgn}(x - c) \left| \dfrac{f(x) - f(c)}{x - c} \right|$, since $f(c) = 0$.

$$g'_+(c) = \lim_{x \to c^+} \text{sgn}(x - c) \left| \frac{f(x) - f(c)}{x - c} \right| = |f'(c)|.$$

$$g'_-(c) = \lim_{x \to c^-} \text{sgn}(x - c) \left| \frac{f(x) - f(c)}{x - c} \right| = -|f'(c)|.$$

Hence g is differentiable at c iff $g'_+(c) = g'_-(c) \quad \Leftrightarrow \quad |f'(c)| = -|f'(c)| \quad \Leftrightarrow \quad f'(c) = 0.$

8. (a) $f(x) = |x| + |x + 1| = \begin{cases} 2x + 1, & \text{for } x \geq 0 \\ 1, & \text{for } -1 \leq x < 0 \\ -2x - 1, & \text{for } x < -1 \end{cases}$

Clearly, $f'(x) = \begin{cases} 2, & \text{for } x > 0 \\ 1, & \text{for } -1 < x < 0 \\ -2, & \text{for } x < -1 \end{cases}$

For $x > 0$, $\dfrac{f(x) - f(0)}{x - 0} = \dfrac{(2x + 1) - 1}{x - 0} = 2 \quad \Rightarrow \quad f'_+(0) = 2$

For $x < 0$, $\dfrac{f(x) - f(0)}{x - 0} = \dfrac{1 - 1}{x - 0} = 0 \quad \Rightarrow \quad f'_-(0) = 0 \neq 2 = f'_+(0).$

Similar procedures proceed for $x < -1, x > -1$.

Hence f is differentiable except $0, -1$.

(b) $g(x) = 2x + |x| = \begin{cases} 3x, & \text{for } x \geq 0 \\ x, & \text{for } x < 0 \end{cases}$

Clearly, $g'(x) = \begin{cases} 3, & \text{for } x > 0 \\ 1, & \text{for } x < 0 \end{cases}$

For $x > 0$, $\dfrac{g(x) - g(0)}{x - 0} = \dfrac{3x - 0}{x - 0} = 3 \quad \Rightarrow \quad g'_+(0) = 3$

For $x < 0$, $\dfrac{g(x) - g(0)}{x - 0} = \dfrac{x - 1}{x - 0} = 1 \quad \Rightarrow \quad g'_-(0) = 1.$

Hence g is differentiable except 0.

(c) $h(x) = x|x| = \begin{cases} x^2, & \text{for } x \geq 0 \\ -x^2, & \text{for } x < 0 \end{cases}$

Clearly, $h'(x) = \begin{cases} 2x, & \text{for } x > 0 \\ -2x, & \text{for } x < 0 \end{cases}$

For $x > 0$, $\dfrac{h(x) - h(0)}{x - 0} = \dfrac{x^2 - 0}{x - 0} = x \quad \Rightarrow \quad h'_+(0) = 0$

For $x < 0$, $\dfrac{h(x) - h(0)}{x - 0} = \dfrac{-x^2 - 0}{x - 0} = -x \quad \Rightarrow \quad h'_-(0) = 0.$

Hence h is differentiable on the whole $\mathbb{R}$.

(d) $k(x) = |\sin x| = \begin{cases} \sin x, & \text{for } \sin x \geq 0 \iff 2n\pi \leq x \leq (2n+1)\pi \\ -\sin x, & \text{for } \sin x < 0 \iff (2n-1)\pi < x < 2n\pi \end{cases}, \forall n \in \mathbb{Z}.$

Clearly, $k'(x) = \begin{cases} \cos x, & \text{for } 2n\pi < x < (2n+1)\pi \\ -\cos x, & \text{for } (2n-1)\pi < x < 2n\pi \end{cases}, \forall n \in \mathbb{Z}.$

For $n \in \mathbb{Z}$ and $x > 2n\pi$, $\dfrac{k(x) - k(2n\pi)}{x - 2n\pi} = \dfrac{\sin x}{x - 2n\pi} = \dfrac{\sin(x - 2n\pi)}{x - 2n\pi} \quad \Rightarrow \quad k'_+(2n\pi) = 1$

For $n \in \mathbb{Z}$ and $x < 2n\pi$, $\dfrac{k(x) - k(2n\pi)}{x - 2n\pi} = \dfrac{-\sin x}{x - 2n\pi} = -\dfrac{\sin(x - 2n\pi)}{x - 2n\pi}$

$\Rightarrow \quad k'_-(2n\pi) = -1$

Similar procedures proceed for $x < (2n+1)\pi, x > (2n+1)\pi, n \in \mathbb{Z}$.

Hence, k is differentiable except $n\pi$ for $n \in \mathbb{Z}$.

9. $f'(-x) = \dfrac{f(-x+h) - f(-x)}{h} = -\lim\limits_{h \to 0} \dfrac{f(x-h) - f(x)}{-h} = -\lim\limits_{h' \to 0} \dfrac{f(x+h') - f(x)}{h'} = -f'(x).$

Hence f' is an odd function.

$g'(-x) = \dfrac{g(-x+h) - g(-x)}{h} = \lim\limits_{h \to 0} \dfrac{[-g(x-h)] - [-g(x)]}{-(-h)} = \lim\limits_{h' \to 0} \dfrac{g(x+h') - g(x)}{h'} = g'(x).$

Hence g' is an even function.

13. Denote $g(h) := \dfrac{f(c+h) - f(c)}{h}$. Hence $\lim\limits_{h \to 0} g(h) = \lim\limits_{h \to 0} \dfrac{f(c+h) - f(c)}{h} = f'(c) \in \mathbb{R}$.

By sequential criterion for limits (Theorem 4.1.8 page 101), denote $h_n := 1/n \neq 0$ for all n, and $\lim h_n = \lim \dfrac{1}{n} = 0$, we have $\lim g(h_n) = \lim\limits_{h \to 0} g(h) = f'(c)$, where

$g(h_n) = \dfrac{f(c + 1/n) - f(c)}{1/n} = n\{f(c + 1/n) - f(c)\}$. Hence $f'(c) = \lim (n\{f(c+1/n) - f(c)\})$.

Take $f(x) := \begin{cases} \sin \pi/x, & x > 0 \\ 0, & x \leq 0. \end{cases}$

At $c = 0$, $n\{f(1/n) - f(0)\} = n(0 - 0) = 0 \ \forall\ n$.

Hence, $\lim (n\{f(c+1/n) - f(c)\}) = 0$.

However, $f'(c)$ doesn't exist because f is not continuous at c.

Or, we may take $f := \chi_{\mathbb{Q}} =$ Dirichlet function. Fix $c \in \mathbb{R}$.

Then $n\{f(c+1/n) - f(c)\} = \begin{cases} n(1-1), & c \in \mathbb{Q} \\ n(0-0), & c \notin \mathbb{Q} \end{cases} = 0 \ \forall\ n.$

The Dirichlet function $\chi_{\mathbb{Q}}$ is not continuous.

Remark If x is rational and y is irrational, why is $x + y$ irrational?

14. Now $h'(x) = 3x^2 + 2 > 0 \ \forall\ x \in \mathbb{R}$. Hence, by Theorem 6.1.8, h^{-1} is differentiable and $(h^{-1})'(y) = \dfrac{1}{h'(x)} = \dfrac{1}{3x^2+2} \quad \forall\ x \in \mathbb{R}$,
where y is related to x by $y = h(x)$.
For $x = 0$, we have $y = h(0) = 1$, and $(h^{-1})'(1) = \dfrac{1}{3(0)+2} = \dfrac{1}{2}$
For $x = 1$, we have $y = h(1) = 4$, and $(h^{-1})'(4) = \dfrac{1}{3(1)+2} = \dfrac{1}{5}$
For $x = -1$, we have $y = h(-1) = -2$, and $(h^{-1})'(-1) = \dfrac{1}{3(1)+2} = \dfrac{1}{5}$.

Supplementary Exercises

1. Consider the function f defined on $[0, \infty)$

$$f(x) = x^\alpha \sin\frac{1}{x}\ , \qquad \alpha > 0\ ,$$

and $f(0) = 0$. Determine the range of α in which

(a) f is continuous on $[0, \infty)$,

(b) f is differentiable on $[0, \infty)$, and

(c) f' exists and is differentiable on $[0, \infty)$.

Solution. This function is smooth, that is, infinitely many times differentiable on $(0, \infty)$. It suffices to consider the case at $x = 0$.

(a) As

$$|x^\alpha \sin\frac{1}{x}| \le x^\alpha,$$

by Sandwich rule

$$\lim_{x\to 0^+} x^\alpha \sin\frac{1}{x} = 0\ ,$$

so f is continuous at $x = 0$ hence we conclude that it is continuous on $[0, \infty)$.

(b) By definition,

$$f'(0) = \lim_{x\to 0^+} \frac{x^\alpha \sin\frac{1}{x} - 0}{x - 0} = \lim_{x\to 0^+} x^{\alpha-1} \sin\frac{1}{x} = 0\ ,$$

when $\alpha > 1$. This limit does not exist when $\alpha \in (0, 1]$. So f is differentiable on $[0, \infty)$ if and only if $\alpha \in (1, \infty)$.

(c) The derivative of f is

$$f'(x) = \alpha x^{\alpha-1} \sin\frac{1}{x} - x^{\alpha-2} \cos\frac{1}{x}\ , \quad x \in (0, \infty)\ ,$$

and $f'(0) = 0$. At $x = 0$, using the definition of the derivative, we have, for $\alpha > 1$,

$$f''(0) = \lim_{x\to 0^+} \frac{\alpha x^{\alpha-1} \sin\frac{1}{x} - x^{\alpha-2} \cos\frac{1}{x} - 0}{x - 0} = \lim_{x\to 0^+} \alpha x^{\alpha-2} \sin\frac{1}{x} - x^{\alpha-3} \cos\frac{1}{x} = 0\ ,$$

when $\alpha \in (3, \infty)$. The limit does not exist when $\alpha \in (0, 3]$. We conclude that f' is differentiable on $[0, \infty)$ if and only if $\alpha \in (3, \infty)$.

2. Find (a) the maximal domain on which the function is well-defined, (b) the domain on which it is continuous and (c) the domain on which it is differentiable in each of the following cases. Justify your answer in (c).

   (a) $f(x) = |x^2 - 5x + 6|$.

   (b) $h(x) = \log(16 - x^2)$.

   (c) $j(x) = \cos|x|$.

   Solution.

   (a) The function is the composition of two functions $f(x) = g(h(x))$ where $h(x) = x^2 - 5x + 6$ and $g(y) = |y|$. Both g and h are continuous on $\mathbb{R}$. As continuity if preserved under composition, f is continuous on $(-\infty, \infty)$.
   Next, write $f(x) = |x^2 - 5x + 6| = |x - 2||x - 3|$. It is known that $x \mapsto |x - 2|$ is not differentiable at 2 and $x \mapsto |x-3|$ is non-zero and differentiable at 2. It follows that f is not differentiable at 2. (See the proposition on next page.) By the same reason f is not differentiable at 3. We conclude that f is differentiable on $(-\infty, 2) \cup (2, 3) \cup (3, \infty)$.

   (b) The function $h = \log(16 - x^2) = \log(k(x))$ where $k(x) = 16 - x^2$ is differentiable everywhere. Using the fact that the log function is defined and smooth only for positive number, h is defined, continuous and differentiable as long as $16 - x^2 > 0$, that is, on $(-4, 4)$.

   (c) j is defined and continuous everywhere. The function $x \mapsto |x|$ is differentiable except at $x = 0$ and $y \mapsto \cos y$ is differentiable everywhere. So j is differentiable at all non-zero x. However, as the derivative of $\cos y$ is equal to 0 at $y = 0$. We must examine the differentiability of j at 0 using definiton. Indeed, using the fact the cosine function is even,

   $$\lim_{h\to 0} \frac{\cos|h| - \cos 0}{h - 0} = \lim_{h\to 0} \frac{\cos h - 1}{h} = 0,$$

   from which we conclude that j is also differentiable at $x = 0$. Hence j is differentiable everywhere.
   A shortcut is to realize that the cosine is an even function, so $j(x) = \cos x$ is differentiable everywhere. In this approach we do not view j as the composite of two functions.

3. Find a function which is not differentiable exactly at the following points on $(-\infty, \infty)$ in each of the following cases:

   (a) n-many distinct points $\{a_1, a_2, \cdots, a_n\}$,

   (b) The set of integers $\mathbb{Z}$, and

   (c) $\left\{0, 1, \frac{1}{2}, \cdots, \frac{1}{n}, \cdots, \right\}$.

   Solution. I forgot to require these functions to be continuous. In the following functions are continuous.

   (a)

   $$f(x) = \sum_{k=1}^{n} |x - a_k| .$$

(b)

$$g(x) = \sum_{k=-\infty}^{\infty} \varphi(x-k),$$

where φ is a function which makes a corner at 0 but otherwise smooth and vanishes outside $[-1,1]$.

(c) You may try this

$$h(x) = \left| x \sin \frac{\pi}{x} \right| .$$

Of course, set $h(0) = 0$.

4. A function $f : (a,b) \to \mathbb{R}$ has a symmetric derivative at $c \in (a,b)$ if

$$f_s'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c-h)}{2h}$$

exists. Show that $f_s'(c) = f'(c)$ if the latter exists. But $f_s'(c)$ may exist even though f is not differentiable at c. Can you give an example?

Solution.

$$\begin{aligned} \frac{f(c+h)-f(c-h)}{2h} &= \frac{f(c+h)-f(c)+f(c)-f(c-h)}{2h} \\ &= \frac{1}{2}\frac{f(c+h)-f(c)}{h} + \frac{1}{2}\frac{f(c-h)-f(c)}{-h} . \end{aligned}$$

Hence we have

$$\begin{aligned} f_s'(c) &= \lim_{h\to 0} \frac{f(c+h)-f(c-h)}{2h} \\ &= \frac{1}{2}\lim_{h\to 0}\frac{f(c+h)-f(c)}{h} + \frac{1}{2}\lim_{h\to 0}\frac{f(c-h)-f(c)}{-h} \\ &= \frac{1}{2}\lim_{h\to 0}\frac{f(c+h)-f(c)}{h} + \frac{1}{2}\lim_{h'\to 0}\frac{f(c+h')-f(c)}{h'} \\ &= \frac{1}{2}f'(c) + \frac{1}{2}f'(c) = f'(c) \end{aligned}$$

Observation. The set-up for $f_s'(c) = \lim_{h\to 0} \frac{f(c+h)-f(c-h)}{2h}$ doesn't involve the value $f(c)$, a simple idea to construct a counter example is by changing the value $f(c)$ from a differentiable function f, so that the new function is not differentiable at c.

Take $f(x) = \begin{cases} 1, & \text{for } x = c \\ 0, & \text{for } x \neq c \end{cases}$. Then $f_s'(c) = \lim_{h\to 0} \frac{f(c+h)-f(c-h)}{2h} = 0$.
But $f'(c)$ doesn't exist since f is not continuous at $x = c$.

5. Let $f : \mathbb{R} \to \mathbb{R}$ satisfy $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$. Suppose f is differentiable at 0 with $f'(0) = 1$. Show that f is differentiable on $\mathbb{R}$ and $f'(x) = f(x)$ for all $x \in \mathbb{R}$.

Solution. If $f \equiv 0$, then $f'(0) = 0 \neq 1$, contradiction arises. Hence $\exists\ x_0 \in \mathbb{R}$ s.t. $f(x_0) \neq 0$.

Then $f(x_0) = f(x_0 + 0) = f(x_0)f(0) \quad \Rightarrow \quad f(0) = 1$.
Also, f is differentiable at 0, hence $\lim_{h\to 0} \frac{f(h)-1}{h} = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = f'(0) = 1$.
Fix x. For all $h \neq 0$, $\frac{f(x+h)-f(x)}{h} = \frac{f(x)f(h)-f(x)}{h} = f(x)\frac{f(h)-1}{h}$
$\Rightarrow \quad f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = f(x) \lim_{h\to 0} \frac{f(h)-1}{h} = f(x)$.
Hence, f is differentiable on $\mathbb{R}$.

The following observation was discussed in class. I formulate it as a proposition below.

Proposition. Let f and g be defined on (a, b) such that f is not differentiable at $c \in (a, b)$ but g is differentiable at c and $g(c) \neq 0$. Then fg is not differentiable at c.

Proof Assume on the contrary that $h(x) = f(x)g(x)$ is differentiable at c. Then $f(x) = \dfrac{h(x)}{g(x)}$ is differentiable at c by the quotient rule, contradiction holds.